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1993 J. Phys. A: Math. Gen. 26 4395

(http://iopscience.iop.org/0305-4470/26/17/044)

J. Phys. A: Math. Gen. **26** (1993) 4395–4405. Printed in the UK

A new truncation in Painlevé analysis

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Received 11 January 1993, in final form 5 May 1993

Abstract. We observe that when the expansion variable in Painlevé analysis satisfies a system of Riccati equations, truncation at a level higher than constant level is allowed. This extends the range of exact solutions to nonlinear partial differential equations that we are able to obtain using truncated Painlevé expansions.

1. Introduction

Weiss, Tabor and Carnevale (WTC) [1] introduced a Painlevé test for partial differential equations (PDEs) which allows the equation in question to be tackled directly. Given a PDE, say

$$U_t = K[U] \tag{1.1}$$

it consists of seeking a solution as an expansion

$$U = \phi^{-\alpha} \sum_{j=0}^{\infty} U_j' \phi^j \tag{1.2}$$

in the neighbourhood of a non-characteristic movable singularity manifold $\phi(x, t) = 0$.

Such an analysis requires first a choice of expansion family, or branch. This is a choice of leading order exponent α, leading order coefficient U_0', and dominant terms $\hat{K}[U]$. For each family there is a set of indices, or resonances, $\mathcal{R} = \{r_1, \ldots, r_n\}$, which give the values of j for which arbitrary data should be introduced in (1.2). We therefore give a choice of family as

$$\alpha,\ U_0',\ \hat{K}[U],\ \beta,\ \mathcal{R} = \{r_1, \ldots, r_n\} \tag{1.3}$$

where β is the weight of $\hat{K}[U]$ when U is of weight α and $\partial/\partial x$ of weight 1.

A simplification of the WTC test has been provided by Conte [2], who distinguishes between the two roles played by the function ϕ, i.e. definition of the singularity manifold and expansion variable. He chooses a new expansion variable,

$$\chi(\phi) = \left(\frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x} \right)^{-1} \tag{1.4}$$

such that the coefficients U_j of the expansion

$$U = \chi^{-\alpha} \sum_{j=0}^{\infty} U_j \chi^j \tag{1.5}$$

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are invariant under the action of the Möbius group on ϕ. (Note that we have used ′ above to denote WTC expansion coefficients).

Defining $S(x,t)$ and $C(x,t)$ to be the homographic invariants

$$S=\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{1.6}$$

$$C=-\frac{\phi_t}{\phi_x} \tag{1.7}$$

the following identities are found to hold:

$$\chi_x=1+\tfrac{1}{2}S\chi^2 \tag{1.8}$$

$$\chi_t=-C+C_x\chi-\tfrac{1}{2}(C_{xx}+CS)\chi^2 \tag{1.9}$$

$$S_t+C_{xxx}+2C_xS+CS_x=0. \tag{1.10}$$

The equation (1.10) is the cross-derivative (integrability) condition of the Riccati system (1.8) and (1.9).

The solution (1.5) is a resummation of (1.2). Transformation formulae between the coefficients of these two expansions are given in [2].

In order to construct a solution with arbitrary data corresponding to *every* index we use a perturbative analysis [3]. This includes the WTC test as a special case. Necessary conditions that a PDE has the Painlevé Property are that, for any family which represents either the general or a particular solution, α be integer (here assumed positive), the indices be distinct integers, and all compatibility conditions corresponding to each index be satisfied.

One of the most important features of the WTC test is the truncation process, and in particular the singular manifold method of Weiss [4]. For example, the Korteweg-de Vries (KdV) equation,

$$U_t=K[U]=(U_{xx}+3U^2)_x \tag{1.11}$$

which has the single family

$$\alpha=2 \qquad U_0'=-2\phi_x^2 \qquad \hat{K}[U]=K[U] \qquad \beta=5 \qquad \mathfrak{R}=\{-1,4,6\} \tag{1.12}$$

admits the truncated expansion [4]

$$U_T=-2\phi_x^2\phi^{-2}+2\phi_{xx}\phi^{-1}+\frac{1}{6}\left(\frac{\phi_t}{\phi_x}\right)-\frac{2}{3}\left(\frac{\phi_{xxx}}{\phi_x}\right)+\frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{1.13}$$

provided that the singular manifold equation (SME)

$$-\frac{\phi_t}{\phi_x}+\left[\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2\right]+\lambda=0 \tag{1.14}$$

holds (here λ is an arbitrary constant).

From this truncation it is possible to obtain the Lax pair for the KdV equation. Similar results hold for most other completely integrable PDEs.

The homographic invariant analysis provides the simplest formalism in which to seek solutions as truncated Painlevé expansions; for the above example the truncation is rewritten

$$U_T=-2\chi^{-2}-\frac{1}{6}(C+4S) \tag{1.15}$$

with SME

$$C+S+\lambda=0. \tag{1.16}$$

The solution U_L of (1.11) which appears in the Lax pair is related to U_T via the classical Darboux transformation

$$U_T - U_L - 2\partial^2 \ln(\psi) = 0 \tag{1.17}$$

where ψ is the eigenfunction of the Lax pair. It is this relationship that is useful in obtaining the Lax pair from truncated Painlevé expansions; an explanation of this procedure can be found in [5].

When the truncation does not lead to a Lax pair, whether or not the equation is completely integrable, it may still be used to find exact solutions (see for example [6–8]). It is this that we concentrate on here. We first derive a new truncation, and then show how this can be used to obtain solutions of PDEs previously unobtainable by truncation.

2. Higher-order truncations

Most authors seem content to use the homographic invariant analysis to rewrite truncations of WTC expansions, as with (1.13) and (1.15) above. However this ignores a new freedom afforded by the change in expansion variable. It is this that we exploit here.

Since we assume the dominant terms of (1.1) to be of weight β, substitution of (1.5) gives

$$K[U] - U_t = \chi^{-\beta} \sum_{j=0}^{\infty} Q_j \chi^j \tag{2.1}$$

for some coefficients Q_j.

If we now consider the Riccati equations (1.8) and (1.9), we see that differentiation of χ^p $(p \neq 0)$ gives terms in χ^{p-1} *and* χ^{p+1}. This means that our leading order analysis (for negative powers of χ) can be mirrored for positive powers. Thus just as when the lowest power of χ in our expansion for U is $-\alpha$ the dominant terms $\hat{K}[U]$ balance and dominate at $X^{-\beta}$, we also have that if the highest power of χ in our expansion for U is α then the same dominant terms $\hat{K}[U]$ will balance and dominate (in positive powers) at χ^{β}.

It then follows that if we consider any pair of expansion families, characterized by (α, U_0, β) and $(\tilde{\alpha}, \bar{U}_0, \bar{\beta})$, we may seek a solution of our PDE as

$$U_T = \chi^{-\alpha} \sum_{j=0}^{\alpha+\hat{\alpha}} U_j \chi^j \tag{2.2}$$

corresponding to which we have

$$K[U] - U_t = \chi^{-\beta} \sum_{j=0}^{\beta+\beta} Q_j \chi^j. \tag{2.3}$$

Since we are assuming (for reasons of simplicity) that the dominant terms of each family depend only on spatial derivatives, it follows that the last coefficient in (2.2) is given by

$$U_{\alpha+\bar{\alpha}} = (-\tfrac{1}{2}S)^{\hat{\alpha}} \bar{U}_0. \tag{2.4}$$

From (2.4) we see that if the truncation (2.2) is genuinely to be at order χ^{α} then we must ask that $S\neq 0$.

We may seek such a solution whenever we use an expansion variable which satisfies a system of Riccati equations, and not just in the particular case where this Riccati system is given by (1.8) and (1.9). If we use the WTC ϕ as an expansion variable then there will be no such balancing of terms at positive powers.

Each of the examples we consider in this paper has the same leading order exponent for all of its families, the latter then being characterized by different choices of U_0. So instead of (2.2) we take

$$U_T=\chi^{-\alpha}\sum_{j=0}^{2\alpha} U_j\chi^j \tag{2.5}$$

this being the form of higher order truncation presented in [9]. The first and last coefficients of (2.5) are determined as

$$U_T=U_0\chi^{-\alpha}+\cdots+(-\tfrac{1}{2}S)^{\alpha}\bar{U}_0\chi^{\alpha} \tag{2.6}$$

where U_0 and $\bar{U}_0$ correspond to any two families of (1.1) (which may of course be the same).

As a simple example, let us consider the KdV equation (1.11). The only solution of (1.11) of the form

$$U_T=\chi^{-2}\sum_{j=0}^{4} U_j\chi^j \qquad U_0U_4\neq 0 \tag{2.7}$$

is

$$U_T=-2\chi^{-2}-\tfrac{1}{6}(C+4S)-\tfrac{1}{2}S^2\chi^2 \tag{2.8}$$

where both S and C are required to be constant. In order to use the results given in the appendix, we write

$$S=-\tfrac{1}{2}k^2 \qquad C=c \tag{2.9}$$

(see (A1) and (A2)). The solution (2.8) then becomes

$$U_T=-\tfrac{1}{2}k^2\{q^2+q^{-2}\}+\tfrac{1}{6}(2k^2-c) \tag{2.10}$$

where q is given by (A6).

In the usual procedure of truncating at constant level, we are able to identify invariant and non-invariant finite expansions. Of course we can still make such an identification; the difference here is that the WTC expansion is still an infinite series. This means that our truncation (2.2) corresponds to the summation of such an expansion.

Using the transformation between invariant and non-invariant analyses [2], we find that (2.8) corresponds to the WTC expansion

$$U=\phi^{-2}\sum_{j=0}^{\infty} U_j'\phi^j \tag{2.11}$$

with the choice of arbitrary data

$$U_4'=-\tfrac{1}{8}k^4\phi_x^{-2} \qquad U_6'=-\tfrac{3}{16}k^4\phi_{xx}^2\phi_x^{-6} \tag{2.12}$$

and subject to the constraints on ϕ

$$C=-\frac{\phi_t}{\phi_x}=c \qquad S=\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2=-\frac{1}{2}k^2. \tag{2.13}$$

Similar remarks hold for all the examples dealt with in this paper.

If we consider the standard truncation (1.15) for the KdV equation, and choose S and C to be given by (2.9), then this gives

$$U_T=-2\chi^{-2}-\tfrac{1}{6}(C+4S)=-\tfrac{1}{2}k^2q^2+\tfrac{1}{6}(2k^2-c). \tag{2.14}$$

Using the identity,

$$\tanh^2(z)+\tanh^{-2}(z)=4\tanh^2\left(2z+\mathrm{i}\frac{\pi}{2}\right)-2 \tag{2.15}$$

the solution (2.10) can be rewritten

$$U_T=-2k^2\tanh^2\left(k(x-ct+F)\right]+\tfrac{1}{6}(8k^2-c) \qquad F=E+\frac{1}{k}\mathrm{i}\frac{\pi}{2} \tag{2.16}$$

a solution of the same form as (2.14). So for this example our higher order truncation yields a different representation, as an infinite WTC expansion, of a solution that may be obtained from the standard truncation.

For this example we have only one family and so must take $U_0=\bar{U}_0$; there is then a natural correspondence between U_0 and $U_{2\alpha}=U_4$. Indeed, we may use the identity (2.15) in reverse, and use the standard truncation (1.15) to obtain the higher-order truncation (2.8). Similar remarks hold for other examples where we choose $U_0=\bar{U}_0$. However, when we have more than one family, it is possible to make a choice of $U_{2\alpha}$ that does not correspond to U_0 in this way. In the next section we show that making such a choice allows us to obtain solutions that cannot be found using truncation at constant level.

3. Examples

3.1. Modified KdV equation. The modified KdV (mKdV) equation,

$$V_t=K[V]=(V_{xx}-2V^3)_x \tag{3.1.1}$$

has two families;

$$\alpha=1 \qquad V_0=\pm 1 \qquad \hat{K}[V]=K[V] \qquad \beta=4 \qquad \mathfrak{R}=\{-1,3,4\}. \tag{3.1.2}$$

Equation (3.1.1) is completely integrable and passes the Painlevé test. The standard truncation is [4]

$$V_T = \pm\chi^{-1} \qquad C+S=0. \tag{3.1.3}$$

We now consider higher-order truncations. The choice $(V_0, \tilde{V}_0) = \pm(1,1)$ gives

$$V_T = \pm(\chi^{-1} - \tfrac{1}{2}S\chi) \tag{3.1.4}$$

where S and C are required to be constant and satisfy $C+4S=0$. Choosing $(V_0, \tilde{V}_0) = \pm(1,-1)$ gives

$$V_T = \pm(\chi^{-1} + \tfrac{1}{2}S\chi) \tag{3.1.5}$$

where S and C are again required to be constant, this time subject to $C-2S=0$.

We assume S and C to be given by (A1) and (A2), and use the identities

$$\tan(z) + \tanh^{-1}(z) = 2\tanh\left(2z + \mathrm{i}\frac{\pi}{2}\right) \tag{3.1.6}$$

$$\tanh(z) - \tanh^{-1}(z) = -2\mathrm{i}\,\mathrm{sech}\left(2z + \mathrm{i}\frac{\pi}{2}\right). \tag{3.1.7}$$

Then (3.1.4) gives

$$V_T = \pm(\chi^{-1} - \tfrac{1}{2}S\chi) = \pm\frac{k}{2}(q+q^{-1}) = \pm k\tanh(k(x-ct+F)) \qquad F = E + \frac{1}{k}\mathrm{i}\frac{\pi}{2} \tag{3.1.8}$$

with $c-2k^2=0$. This may also be obtained from (3.1.3). However (3.1.5) gives

$$V_T = \pm(\chi^{-1} + \tfrac{1}{2}S\chi) = \pm\frac{k}{2}(q-q^{-1}) = \mp\mathrm{i}k\,\mathrm{sech}(k(x-ct+F) \qquad F = E + \frac{1}{k}\mathrm{i}\frac{\pi}{2} \tag{3.1.9}$$

with $c+k^2=0$, and so we recover the one-soliton solution of the mKdV equation. This cannot be obtained from the constant level truncation.

The Miura transformation,

$$U = -V_x - V^2, \tag{3.1.10}$$

maps the solutions (3.1.9) onto truncations representing the one-soliton solution of the KdV equation. This is not true of either of the solutions (3.1.8).

In the rest of our examples we will concentrate on higher-order truncations having $U_0 \neq \tilde{U}_0$. The solutions thus obtained for these equations are presented in table 1.

3.2. *Kolmogoroff–Petrovsky–Piscounov equation*

The Kolmogoroff–Petrovsky–Piscounov (KPP) equation [10],

$$U_t = K[U] = U_{xx} - 2U^3 + U \tag{3.2.1}$$

has two families;

$$\alpha=1 \qquad U_0=\pm 1 \qquad \hat{K}[U] = U_{xx} - 2U^3 \qquad \beta=3 \qquad \mathcal{R} = \{-1, 4\}. \tag{3.2.2}$$

The index at 4 produces the compatibility condition [6, 7]

$$C(C_t + CC_x - \tfrac{1}{3}C^3 + \tfrac{3}{2}C) = 0 \tag{3.2.3}$$

Table 1. Solutions from higher-order truncation which cannot be obtained from standard truncation.

Equation	α	$(U_0, \hat{U}_0)$	Truncation	Solution
mKdV	1	$\pm(1,-1)$	$V_T=\pm(\chi^{-1}+\frac{1}{2}S\chi)$ S, C constant; $C-2S=0$	$V_T=\mp ik\,\mathrm{sech}(k(x-ct+F))$ $c+k^2=0$
KPP	1	$(-1,1)$	$U_T=-\chi^{-1}-\frac{1}{2}S\chi$ $C=O, S=\frac{1}{2}$	$U_T=ik\,\mathrm{sech}(k(x+F))$ $k^2+1=0$
Nepomnyachtchyi	1	$(\nu,-\nu)$	$U_T=\nu\chi^{-1}+\frac{1}{2}S\nu X$ $d=\gamma=0,\ C=0,\ S=a/2b$	$U_T=-ik\nu\,\mathrm{sech}(k(x+F))$ $d=\gamma=0,\ -\frac{1}{2}k^2=a/2b$
'Bretherton'	2	$(2,-2)$	$U_T=2\chi^{-2}+\frac{1}{30}(C^2+2OS+1)-\frac{1}{2}S^2\chi^2$ $S=-\frac{1}{20}(C^2+1),\ 11(C^2+1)^2+100=0$ (S, C assmed constant)	$U_T=\{2ik\,\mathrm{sech}(k(x-ct+F))\}_x$ $c^2=10k^2-1,\ 11k^4+1=0$
Chazy class VII	1	$(3,-1)$	$U_T=3\chi^{-1}+\frac{1}{2}S\chi$ $S_x=0$	$U_T=k\tanh(k(x+F))-2ik\,\mathrm{sech}(k(x+F))$
Potential KdV7	1	$(6,2)$	$W_T=6\chi^{-1}+\mu+(\frac{7}{2}S+\omega)x$ $+\frac{2240}{109503}\omega^3(12033S+4517\omega)t-S\chi$ $69S^2+60S\omega+20\omega^2=0,$ $C=-\frac{1}{2}(257S^3+882S^2\omega+140S\omega^2+280\omega^3)$ ω, μ arbitrary constants	$W_T=4k\tanh(k(x-ct+F))-2ik\,\mathrm{sech}(k(x-ct+F))$ $+\mu+(-\frac{7}{4}k^2+\omega)x+\frac{2240}{109503}\omega^3\left(-\frac{12033}{2}k^2+4517\omega\right)t$ $69k^4-120\omega k^2+80\omega^2=0,$ $c=-\frac{80}{1587}\omega^2(2515k^2+884\omega)$

and (3.2.1) does not have the Painlevé property. Solutions of (3.2.1) obtained by truncation at constant level are given in [6, 7].

Higher-order truncation with the choice $(U_0, \tilde{U}_0)=(-1,1)$ gives a (stationary) sech solution; see table 1. For this solution the compatibility condition (3.2.3) is satisfied.

3.3. *Nepomnyachtchyi equation*

The Nepomnyachtchyi equation [11],

$$U_t+K[U]=U_t+aU_{xx}+bU_{xxxx}-2b\nu^{-2}(U^3)_{xx}+\gamma UU_x+dU=0 \qquad b\nu\neq 0 \qquad (3.3.1)$$

has the families

$$\alpha=1 \qquad U_0=\pm\nu \qquad \hat{K}[U]=bU_{xxxx}-2b\nu^{-2}(U^3)_{xx}$$
$$\beta=5 \qquad \mathfrak{R}=\{-1,3,4,4\}. \qquad (3.3.2)$$

The double index means that the equation does not have the Painlevé property. The compatibility conditions at 3 and 4 are [6]

$$\nu C=0 \qquad \nu(d-C_x)=0 \qquad (3.3.3)$$

respectively.

Seeking a higher-order truncation with $(U_0, \tilde{U}_0)=(\nu, \nu)$ gives the tanh solution presented in [6]. Choosing $(U_0, \tilde{U}_0)=(\nu, -\nu)$ gives a sech solution (see table 1), for which the compatibility conditions (3.3.3) are satisfied.

Note that autonomous PDEs can have pure sech solutions only when the equation is invariant under $U\to -U$.

3.4. *Bretherton equation*

The 'Bretherton equation' (considered by Kudryashov [8]),

$$U_{tt}+K[U]=U_{tt}+U_{xxxx}-30U^3+U_{xx}+U=0 \qquad (3.4.1)$$

has the families;

$$\alpha=2 \qquad U_0\pm 2 \qquad \hat{K}[U]=U_{xxxx}-30U^3 \qquad \beta=6$$
$$\mathfrak{R}=\left\{-1,8,\frac{7\pm i\sqrt{71}}{2}\right\}. \qquad (3.4.2)$$

The complex indices mean that (3.4.1) does not have the Painlevé property. If we ignore these indices we still have a compatibility condition at 8; this is identically satisfied.

For reasons of simplicity we assume S and C are constant. The choice $(U_0\tilde{U}_0)=(2,2)$ is equivalent to the constant level truncation considered in [8], and gives a solution quadratic in tanh. The choice $(U_0\tilde{U}_0)=(2,-2)$ gives a derivative sech solution (see table 1), which is a travelling wave of fixed speed.

3.5. *Chazy class VII*

All of the examples that we have considered so far have had two families which correspond to the invariance of the dominant terms under $U\to -U$. We now consider a simple example, an ordinary differential equation (ODE), where this is no longer the case.

Chazy class VII [12] is the set of third-order ODEs of the form U_{xxx} polynomial in U_{xx}, U_x and U, with coefficients analytic in x, whose dominant terms are given by

$$K[U]=U_{xxx}-UU_{xx}-2U_x^2-2U^2U_x=0. \tag{3.5.1}$$

They have the following families;

$$\alpha=1 \qquad U_0-1 \qquad \hat{K}[U]=K[U] \qquad \beta=4 \qquad \mathfrak{R}=\{-1,2,4\} \tag{3.5.2}$$

$$\alpha=1 \qquad U_0=3 \qquad \hat{K}[U]=K[U] \qquad \beta=4 \qquad \mathfrak{R}=\{-1,4,6\} \tag{3.5.3}$$

The ODE (3.5.1) is, up to Möbius transformations, the only equation in this class with the Painlevé property. We now use the truncation process to obtain particular solutions of (3.5.1).

Truncation at constant level gives the solutions;

$$U_T=-\chi^{-1} \qquad S_x=0 \tag{3.5.4}$$

$$U_T=3\chi^{-1} \qquad S=0. \tag{3.5.5}$$

The first of these is non-trivial, giving

$$U_T=-\frac{k}{2}\tanh\left(\frac{k}{2}(x+E)\right). \tag{3.5.6}$$

To obtain different solutions from higher-order truncations we must choose $U_0 \neq \bar{U}_0$. The choices $(U_0, \bar{U}_0)=(-1,3)$, $(3,-1)$ give equivalent truncations; in table 1 we take $(U_0, \bar{U}_0)=(3,-1)$.

In rewriting the truncation given in table 1 we have used the identities

$$q=\tanh(k(x-ct+F))-\mathrm{i}\,\mathrm{sech}(k(x-ct+F)) \tag{3.5.7}$$

and

$$q^{-1}=\tanh(k(x-ct+F))+\mathrm{i}\,\mathrm{sech}(k(x-ct+F)) \tag{3.5.8}$$

(see (3.1.8) and (3.1.9)). We remark that recently Conte and Musette [13] have proposed a method of obtaining solitary wave solutions of nonlinear PDEs as polynomials in two functions σ and τ. These functions have a dependence on a parameter μ such that when $\mu=0$ $\sigma=\mathrm{i}\,\mathrm{sech}(k(x-ct+F))$ and $\tau=\tanh(k(x-ct+F))$. From (3.5.7) and (3.5.8) we see that the analysis presented in [13] is then related to that given here through

$$q=\tau-\sigma \qquad q^{-1}=\tau+\sigma. \tag{3.5.9}$$

The advantage of our approach is that it is placed within the context of Painlevé analysis, and the well understood derivation of exact solutions via truncation.

3.6. Potential seventh-order KdV equation

The potential seventh-order KdV equation (potential KdV7),

$$W_t = K[W] = W_{7x} + 14W_xW_{5x} + 28W_{2x}W_{4x} + 21W_{3x}^2 + 70W_x^2W_{3x} + 70W_xW_{2x}^2 + 35W_x^4, \tag{3.6.1}$$

is completely integrable and passes the Painlevé test. This has the families;

$$\alpha=1 \qquad W_0=2 \qquad \hat{K}[W]=K[W] \qquad \beta=8$$
$$\mathfrak{R}=\{-1,1,2,4,5,7,10\} \tag{3.6.2}$$
$$\alpha=1 \qquad W_0=6 \qquad \hat{K}[W]=K[W] \qquad \beta=8$$
$$\mathfrak{R}=\{-3,-1,1,2,7,10,12\} \tag{3.6.3}$$
$$\alpha=1 \qquad W_0=12 \qquad \hat{K}[W]=K[W] \qquad \beta=8$$
$$\mathfrak{R}=\{-5,-3,-1,1,10,12,14\}. \tag{3.6.4}$$

We consider only the construction of higher-order truncations having $W_0 \neq \bar{W}_0$. Any such choice with either W_0 or $\bar{W}_0$ being 12 requires $S=0$, and so is discounted. Thus only the choice $(W_0, \bar{W}_0)=(6, 2)$ appears in table 1; choosing $(W_0, \bar{W}_0)=(2, 6)$ gives an equivalent truncation.

A solution of KdV7 is obtained from a solution of (3.6.1) by $U=W_x$. We note however that there is no truncation

$$V_T = V_0\chi^{-1} + V_1 + V_2\chi \tag{3.6.5}$$

of the seventh-order mKdV equation such that the solution $U=(W_T)_x$ of KdV7, W_T as in table 1, can be obtained using the Miura map (3.1.10).

4. Conclusions

When the expansion variable in Painlevé analysis satisfies a system of Riccati equations we are able to truncate at a level higher than constant level. This is a further advantage of the invariant analysis over the non-invariant analysis. Such truncations correspond to the summation of infinite WTC expansions for certain choices of arbitrary data.

Appendix

Here we show how a solution U_T, obtained as a truncated Painlevé expansion with coefficients given in terms of S and C, is determined by the equations

$$C=c \tag{A1}$$
$$S=-\tfrac{1}{2}k^2 \tag{A2}$$

c and k being arbitrary constants.

When performing a Painlevé analysis using the homographic invariant formalism no reference need be made to the definition of χ given by (1.4). We only need grad(χ) given by (1.8) and (1.9) together with the cross-derivative condition (1.10). This remains true when determining U_T; we do not need to solve (A1), (A2) for ϕ.

When (A1) and (A2) hold, the condition (1.10) is satisfied, and (1.8) and (1.9) tell us

$$\chi_x = 1 - \tfrac{1}{4}k^2\chi^2 \tag{A3}$$

$$\chi_t = -c\chi_x. \tag{A4}$$

The general solution of (A3) and (A4) can be written

$$\chi = \frac{2}{k}q^{-1} \tag{A5}$$

where

$$q(x, t) = \tanh\left(\frac{k}{2}(x - ct + E)\right) \tag{A6}$$

and E is an arbitrary constant.

The choice q^{-1} in (A5) is made simply so that standard truncations give expressions polynomial in tanh.

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